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TRANSFORMATIONS USEFUL IN LINEAR BETATRON THEORY

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PURPOSE

To find a contact transformation in two degrees of freedom that effects the Courant-Snyder transformation simultaneously for both degrees of freedom. Subsequently, to find a contact transformation that transforms to action-angle variables in two degrees of freedom.

EQUATIONS OF MOTION

Betatron motion in the linear approximation with median plane symmetry is uncoupled and given by:

$$\frac{d^2 q_x}{d\,s^2} + K_x(s)\; q_x = 0 \tag{1}$$

and

$$\frac{d^2 q_y}{d\,s^2} + K_y(s)\; q_y = 0 \tag{2}$$

CONTACT TRANSFORMATION

Equations (1) and (2) may be derived from the hamiltonian

$$H = \frac{1}{2}\, p_x^2 + \frac{1}{2}\, K_x\, q_x^2 + \frac{1}{2}\, p_y^2 + \frac{1}{2}\, K_y\, q_y^2\ , \tag{3}$$

where (p_x, q_x) and (p_y, q_y) are canonically conjugate variables for each degree of freedom.

To execute the first transformation, choose a linear transformation represented by the following generator

$$F_2\,(P_x, q_x, P_y, q_y; s) = \frac{1}{\sqrt{\beta_x}}\, P_x\, q_x + \frac{\beta_x'}{4\beta_x}\, q_x^2$$

$$+ \frac{1}{\sqrt{\beta_y}}\, P_y\, q_y + \frac{\beta_y'}{4\beta_y}\, q_y^2\,, \tag{4}$$

where (P_x, Q_x) and (P_y, Q_y) are the new canonical momenta and coordinates. Also β_x and β_y are the periodic functions of s introduced by Courant and Snyder[1] which possess the properties:

$$\frac{1}{2}\,\beta_x''\,\beta_x - \frac{1}{4}\,\beta_x'^2 + \beta_x^2\,K_x = 1\,, \tag{5}$$

and

$$\frac{1}{2}\,\beta_y''\,\beta_y - \frac{1}{4}\,\beta_y'^2 + \beta_y^2\,K_y = 1\,. \tag{6}$$

Equation (4) results in the coordinate transformation

$$p_x = \frac{1}{\sqrt{\beta_x}}\,(P_x + \frac{1}{2}\,\beta_x'\,Q_x) \tag{7}$$

$$q_x = \sqrt{\beta_x}\,Q_x \tag{8}$$

$$p_y = \frac{1}{\sqrt{\beta_y}}\,(P_y + \frac{1}{2}\,\beta_y'\,Q_y) \tag{9}$$

$$q_y = \sqrt{\beta_y}\,Q_y\,, \tag{10}$$

and the new hamiltonian becomes

$$K = H + \frac{\partial F_2}{\partial s} = \frac{1}{2\beta_x}\,(P_x^2 + Q_x^2) + \frac{1}{2\beta_y}\,(P_y^2 + Q_y^2)\,. \tag{11}$$

To see that the customary equations of motion result, one finds from Eq. (11)

$$\frac{dQ_x}{ds} = \frac{P_x}{\beta_x}\quad,\quad \frac{dQ_y}{ds} = \frac{P_y}{\beta_y}\,, \tag{12}$$

and

$$\frac{dP_x}{ds} = -\frac{Q_x}{\beta_x} \quad , \quad \frac{dP_y}{ds} = -\frac{Q_y}{\beta_y} \; . \tag{13}$$

Then, if one introduces:

$$\xi = \frac{1}{\nu_x} \int \frac{ds}{\beta_x} \quad , \quad \eta = \frac{1}{\nu_y} \int \frac{ds}{\beta_y} \tag{14}$$

as is done by Courant and Snyder[1], one has

$$\frac{d^2 Q_x}{d\xi^2} + \nu_x^2 Q_x = 0 \quad , \tag{15}$$

and

$$\frac{d^2 Q_y}{d\eta^2} + \nu_y^2 Q_y = 0 \; . \tag{16}$$

ACTION-ANGLE VARIABLES

Finally, one may remove the s-dependence of the hamiltonian in Eq. (11) by transforming to the action-angle variables $(\rho_x, \phi_x, \rho_y, \phi_y)$. Consider first only one degree of freedom. Take the generator of the form

$$F_1(Q,\phi;s) = \frac{1}{2} Q^2 f(\phi,s). \tag{17}$$

Then

$$P = Qf,$$

$$\rho = -\frac{1}{2} Q^2 \frac{\partial f}{\partial \phi} \; , \tag{18}$$

and the new hamiltonian is

$$W = \rho \frac{\left(\frac{f^2+1}{\beta} + \frac{\partial f}{\partial s}\right)}{\left(-\frac{\partial f}{\partial \phi}\right)} \; . \tag{19}$$

Since the desire is to have W independent of s, choose $f(\phi,s)$ such that

$$\frac{f^2+1}{\beta} + \frac{\partial f}{\partial s} + C\,\frac{\partial f}{\partial \phi} = 0. \tag{20}$$

The characteristics of Eq. (20) are solutions of

$$\frac{ds}{\beta} = \frac{d\phi}{C\beta} = -\,\frac{df}{1+f^2} \tag{21}$$

which gives

$$\phi - Cs = C_1 \; ; \; \mathrm{Cot}^{-1} f - \int \frac{ds}{\beta} = C_2 \; . \tag{22}$$

A general solution of Eq. (20) is then

$$f(\phi,s) = \mathrm{Cot}\left[g(\phi - Cs) + \int \frac{ds}{\beta}\right] \; , \tag{23}$$

where g is an arbitrary function. Any definite choice of g selects a particular canonical set (ρ,ϕ) that is suitable as an action-angle pair. To see that this is true, let (ρ_1,ϕ_1) be the pair generated by F_1 of Eqs. (17) and (22). Transform to the pair (ρ_2,ϕ_2) using the generator

$$F_2(\rho_2,\phi_1,s) = \rho_2\left[Cs + g(\phi_1 - Cs)\right] \; . \tag{24}$$

Then

$$\rho_1 = \rho_2 g'(\phi_1 - Cs) \; , \tag{25}$$

$$\phi_2 = Cs + g(\phi_1 - Cs) \; , \tag{26}$$

and the new hamiltonian

$$U = C\,\rho_2 g' + \rho_2\left[C - Cg'\right] = C\rho_2 \; . \tag{27}$$

But, from Eqs. (17) and (22)

$$P = \sqrt{\frac{2\rho_1}{g'}} \text{ Cos } \left(g + \int \frac{ds}{\beta}\right) \tag{28}$$

$$Q = \sqrt{\frac{2\rho_1}{g'}} \text{ Sin } \left(g + \int \frac{ds}{\beta}\right), \tag{29}$$

which, using Eqs. (25) and (26) becomes

$$P = \sqrt{2\rho_2} \text{ Cos } \left(\phi_2 - Cs + \int \frac{ds}{\beta}\right) \tag{30}$$

$$Q = \sqrt{2\rho_2} \text{ Sin } \left(\phi_2 - Cs + \int \frac{ds}{\beta}\right). \tag{31}$$

Thus there is no loss of generality in choosing

$$g = \phi - Cs \tag{32}$$

in Eq. (22).

The relation of the constant C to the tune ν may be found by noticing that, since the hamiltonian is

$$W = C\rho\ , \tag{33}$$

$d\phi/ds = C$ or $2\pi\nu$ = Cx circumference. If the circumference is written as $2\pi R$ where R is the average radius

$$C = \frac{\nu}{R}\ . \tag{34}$$

Thus

$$W = \frac{\nu}{R}\rho\ . \tag{35}$$

With two degrees of freedom the corresponding generator becomes

$$F_1(Q_x,\phi_x,Q_y,\phi_y;s) = \frac{1}{2} Q_x^2 \text{ Cot } \psi_x + \frac{1}{2} Q_y^2 \text{ Cot } \psi_y, \tag{36}$$

where

$$\psi_x = \phi_x - \frac{\nu_x}{R} s + \int \frac{ds}{\beta}\ , \tag{37}$$

and

$$\psi_y = \phi_y - \frac{\nu_y}{R} s + \int \frac{ds}{\beta} . \qquad (38)$$

This generator results in the transformation

$$P_x = \sqrt{2\rho_x} \, \text{Cos } \psi_x , \qquad (39)$$

$$Q_x = \sqrt{2\rho_x} \, \text{Sin } \psi_x , \qquad (40)$$

$$P_y = \sqrt{2\rho_y} \, \text{Cos } \psi_y , \qquad (41)$$

$$Q_y = \sqrt{2\rho_y} \, \text{Sin } \psi_y , \qquad (42)$$

and the new hamiltonian becomes

$$W = \frac{\nu_x}{R} \rho_x + \frac{\nu_y}{R} \rho_y \quad . \qquad (43)$$

In summary the hamiltonian representing linear betatron motion with median plane symmetry has been successively transformed with the generators F_2 and F_1 to yield the hamiltonian in Eq. (43) which is independent of s. The overall transformation of the momenta and coordinates is

$$p_x = \sqrt{\frac{2\rho_x}{\beta x}} \left(\text{Cos } \psi_x + \frac{\beta_x'}{2} \text{ Sin } \psi_x \right) , \qquad (44)$$

$$q_x = \sqrt{2\beta_x \rho_x} \, \text{Sin } \psi_x , \qquad (45)$$

$$p_y = \sqrt{\frac{2\rho_y}{\beta y}} \left(\text{Cos } \psi_y + \frac{\beta_y'}{2} \text{ Sin } \psi_y \right) , \qquad (46)$$

$$q_y = \sqrt{2\beta_y \rho_y} \, \text{Sin } \psi_y \quad . \qquad (47)$$

The hamiltonian W, being independent of s is an invariant. This invariant expressed in terms of the original canonical momenta and coordinates is

$$W = \frac{\nu_x}{2R\beta_x}\left\{q_x^2 + \left(-\frac{\beta_x'}{2}\,q_x + \beta_x\,p_x\right)^2\right\}$$
$$+ \frac{\nu_y}{2R\beta_y}\left\{q_y^2 + \left(-\frac{\beta_y'}{2}\,q_y + \beta_y p_y\right)^2\right\}. \tag{48}$$

Since $p_x = q_x'$ and $p_y = q_y'$, Eq. (48) is seen to have the form given by Courant and Snyder[1]. In fact, as pointed out by Courant[2], the transformation from H to W which gives rise to the momenta and coordinate transformation of Eqs. (44) to (47) may be accomplished by a single generator.

$$F_1\,(q_x,\ \phi_x,\ q_y,\ \phi_y;\ s) = \frac{q_x^2}{2\beta_x}\,(\cot\,\psi_x + \tfrac{1}{2}\,\beta_x')$$
$$+ \frac{q_y^2}{2\beta_y}\,(\cot\,\psi_y + \tfrac{1}{2}\,\beta_y'). \tag{49}$$

REFERENCES

1. E. D. Courant and H. S. Snyder, Annals of Physics 3, 1 (1958).
2. E. D. Courant, private communication.